Computational Models - Lecture 3[1]

Handout Mode

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[1] Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University.

Lecture highlight

תן לי הכוח לשנות את מה שאני יכול לשנות, לקבל את מה שאינני
יכול לשנות, ואת החוכמה להבדיל בין שניהם.

Computational Models - Lecture 3

- Non-regular languages: two approaches
  1. Pumping Lemma
  2. Myhill-Nerode Theorem (not in Sipser's book)
- Closure properties
- Algorithmic questions for NFAs

- Sipser, 1.4, 2.1, 2.2
- Hopcroft and Ullman, 3.4

Proved last time

Theorem 1

A language is described by a regular expression, iff it is regular.

We have made a lot of progress understanding what finite automata can do, but what they cannot do?

Negative results

Is there a DFA that accepting the following languages (over $\{0,1\}$).

- $\mathcal{B} = \{0^n 1^n : n \geq 0\}$
- $\mathcal{C} = \{w : \#_1(w) = \#_0(w)\}$
- $\mathcal{D} = \{w : \#_{01}(w) = \#_{10}(w)\}$

$\#_s(w)$ – the number of times s appears in w.

Consider $\mathcal{B}$:

- DFA must "remember" how many 0's it has seen
- Impossible with finite state.

The others languages seem to be exactly the same...

Question: Is this a proof?

Answer: No, $\mathcal{D}$ is regular.....

Part I

Pumping Lemma

Regular languages can be pumped

For every regular language $\mathcal{L}$, there exists $\ell > 0$, the pumping length, s.t.:
Every $s \in \mathcal{L}$ longer than ℓ, can be "pumped" into a longer string in $\mathcal{L}$.

This is a powerful technique for showing that a language is not regular.

The Pumping Lemma

Lemma 2

For every ***regular*** *language* $\mathcal{L}$*, exists* $\ell > 0$ *(i.e., the pumping length)* $s.t.$*: every* $s \in \mathcal{L}$ *with* $|s| \geq \ell$*, can be written as* $s = xyz$ *with*

1. $xy^i z \in \mathcal{L}$ *for every* $i \geq 0$*,*
2. $|y| > 0$*, and*
3. $|xy| \leq \ell$*.*

Remarks: Without the second condition, the theorem would be trivial.

The third condition is technical and sometimes useful.

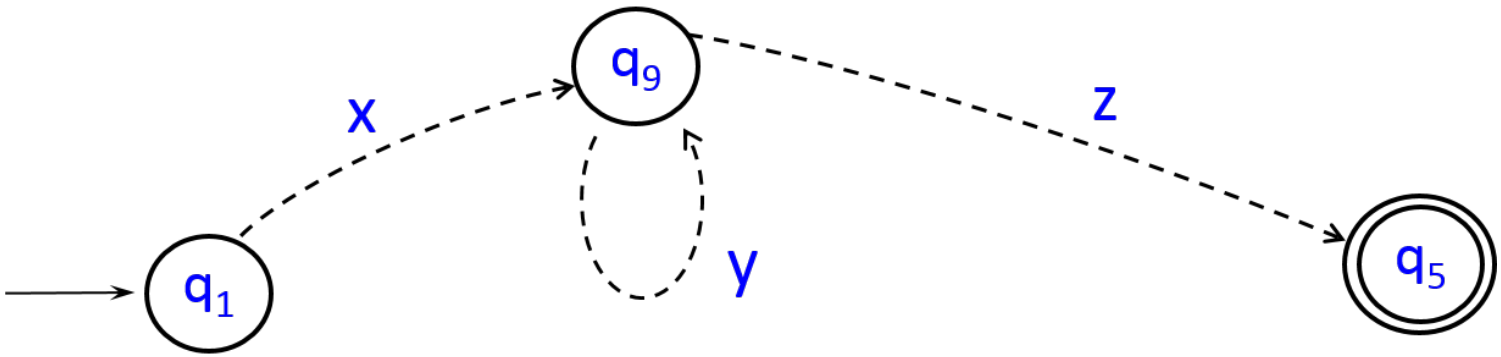
Proving the Pumping Lemma

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA accepting $\mathcal{L}$, and let $\ell = |Q|$.

Let $s \in \mathcal{L}$ be with $|s| \geq \ell$, and consider the sequence of states M traverse as it reads $s = s_1 \dots s_n$:

	s_1		s_2		s_3		s_4		s_5		s_6		$\dots$		s_n	
$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$
q_1		q_{20}		q_9		q_{17}		q_{12}		q_{13}		q_9		q_2		$q_5 \in F$

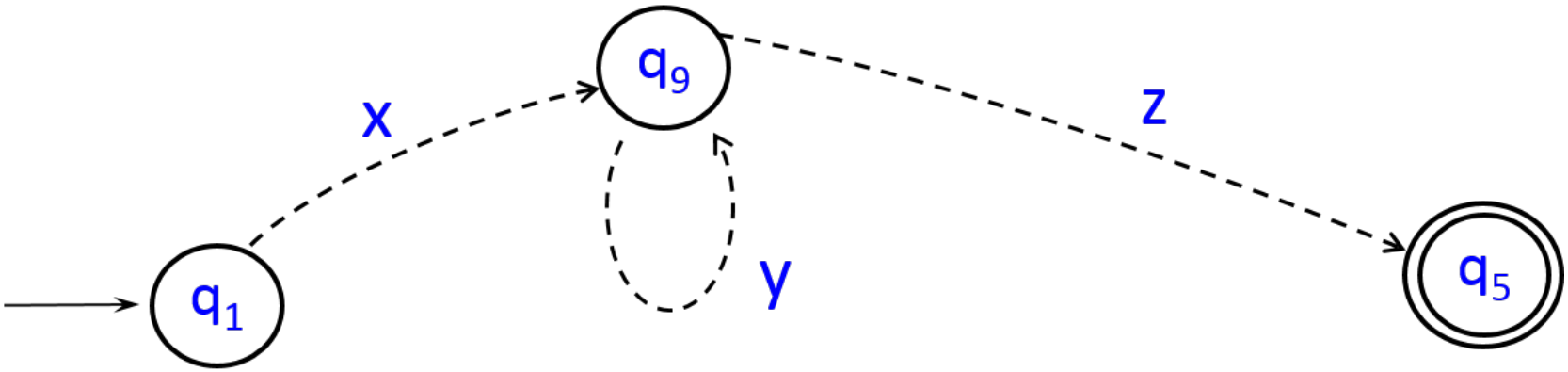
By the pigeonhole principle, at least one of the states in the above sequence repeats.



$s = xyz$

Proving the Pumping Lemma, cont.

$s = xyz$



- By inspection, M accepts $xy^k z$ for *every* $k \geq 0$.
- $|y| > 0$, because the state q_9 is repeated.
- To ensure that $|xy| \leq \ell$, pick first state repetition, which must occur no later than $\ell + 1$ states in sequence.

Application # 1

Corollary 3

$\mathcal{B} = \{0^n 1^n : n > 0\}$ *is not regular.*

Proof: By contradiction. Suppose $\mathcal{B}$ is regular and let ℓ be its pumping length.

- Consider the string $s = 0^\ell 1^\ell \in \mathcal{B}$.
- Let x, y, z be (one possible) strings guaranteed by the pumping lemma (i.e., $s = xyz$)
  1. $xy^i z \in \mathcal{B}$ for *every* $i \geq 0$,
  2. $|y| > 0$, and
  3. $|xy| \leq \ell$.
- If y is all 0, then xy^2z has too many 0's.
- If y is all 1, then xy^2z has too many 1's.
- If y is mixed, then xy^2z is not of right form. □

We did not use the third property.

Application # 2

Corollary 4

$\mathcal{C} = \{w : \#_1(w) = \#_0(w)\}$ *is not regular.*

Proof: By contradiction. Suppose $\mathcal{C}$ is regular. Let ℓ be the pumping length.

- Consider the string $s = 0^\ell 1^\ell \in \mathcal{C}$.
- Let x, y, z be (possible) set of strings guaranteed by the pumping lemma (i.e., $s = xyz$)
  1. $xy^i z \in \mathcal{B}$ for *every* $k \geq 0$,
  2. $|y| > 0$, and
  3. $|xy| \leq \ell$.
- Since $|xy| \leq \ell$, the string y is all 0's.
- Thus, $xy^2 z \notin \mathcal{C}$ (more 0's than 1's). □

Could we have used $s = (01)^\ell$?

Application # 3

Corollary 5

$\mathcal{E} = \{0^i 1^j : i > j\}$ *is not regular.*

Proof: By contradiction. Suppose $\mathcal{E}$ is regular. Let ℓ be its pumping length.

- Consider the string $s = 0^\ell 1^{\ell-1} \in \mathcal{E}$.
- By pumping lemma, $s = xyz$, where $xy^k z \in \mathcal{E}$ for every $k \geq 0$, $|y| > 0$ and $|xy| \leq \ell$.
- But $xy^0 z = xz \notin \mathcal{E}$ (at least as much 1's as 0's) □

Application # 4

Corollary 6

The language $Primes \subset 1^*$ – all strings whose length is a prime number – is not regular.

Proof: Suppose $Primes$ is regular, and let ℓ be its pumping length.

- Let $s = 1^p \in Primes$, where $p \geq \ell$ is a prime (?)
- By pumping lemma, $s = xyz$, where $xy^k z \in Primes$ for every $k \geq 0$.
- Let $|y| = m$. Hence, $xy^{p+1}z = 1^{p+mp} \in Primes$

  but $p(m+1)$ is not prime... □

Another Example

Consider the language $\mathcal{L} = \{a^i b^n c^n : n \geq 0, i \geq 1\} \cup \{b^n c^m : n, m \geq 0\}$.

Any non-empty $s \in \mathcal{L}$ can be pumped:

- If $s = a^i b^n c^n$ with $i > 0$, then set $x = \varepsilon$ and $y = a$.
- If $s = b^n c^m$ with $n > 0$, then set $x = \varepsilon$ and $y = b$.
- If $s = c^m$ with $m > 0$, then set $x = \varepsilon$ and $y = c$.

  (in all cases z is set arbitrarily).

- Is $\mathcal{L}$ regular? No
- How can we prove it?

Part II

Characterization of Regular Languages

The equivalence relation $\overset{\mathcal{L}}{\sim}$

Definition 7

For $\mathcal{L} \subseteq \Sigma^*$, define the equivalence relation $\overset{\mathcal{L}}{\sim}$ over words in Σ^*, by $x \overset{\mathcal{L}}{\sim} y$ if for every $z \in \Sigma^*$, it holds that $xz \in \mathcal{L} \iff yz \in \mathcal{L}$.

Easy to see that $\overset{\mathcal{L}}{\sim}$ is indeed an equivalence relation (reflexive, symmetric, transitive) on Σ^*.
Hence, $\overset{\mathcal{L}}{\sim}$ partitions Σ^* into equivalence classes.

For $x \in \Sigma^*$, let $[x] \subseteq \Sigma^*$ denote its equivalence class with respect to $\overset{\mathcal{L}}{\sim}$

How many equivalence classes does $\overset{\mathcal{L}}{\sim}$ induce? finite or infinite?

Could be either (depends on $\mathcal{L}$).

Fact 8 (right invariance)

If $x \overset{\mathcal{L}}{\sim} y$, then $xw \overset{\mathcal{L}}{\sim} yw$ for every $w \in \Sigma^*$

Proof: $(xw)z \in \mathcal{L} \leftrightarrow x(wz) \in \mathcal{L} \leftrightarrow y(wz) \in \mathcal{L} \leftrightarrow (yw)z \in \mathcal{L}$ □

Three examples

- $\mathcal{L}_1 = \{w : \#_1(w) \bmod 5 = 0\}$

  $\overset{\mathcal{L}_1}{\sim}$ has finitely many equivalence classes.

  The equivalent classes are: $[\varepsilon]$, $[1]$, $[11]$, $[111]$, $[1111]$.

  Proof:

  - Classes cover $\{0,1\}^*$: for any $x \in \{0,1\}^*$: $x \overset{\mathcal{L}}{\sim} 1^{\#_1(w) \bmod 5}$.
    $xz \in \mathcal{L} \iff \#_1(xz) \bmod 5 = 0 \iff$
    $(\#_1(x) \bmod 5) + \#_1(z) \bmod 5 = 0 \iff 1^{\#_1(x) \bmod 5} z \in \mathcal{L}$.
  - Classes are disjoint: $1^i \overset{\mathcal{L}}{\not\sim} 1^j$ for $i \neq j \in \{0,1,2,3,4\}$

- $\mathcal{L}_2 = \{0^n 1^n : n \in \mathbb{N}\}$

  $\overset{\mathcal{L}_2}{\sim}$ has infinitely many equivalence classes.

  $[0] \neq [0^2] \neq [0^3] \ldots$

- $\mathcal{L}_3 = \{a^i b^n c^n : n \geq 0, i \geq 1\} \cup \{b^n c^m : n, m \geq 0\}$

  $\overset{\mathcal{L}_3}{\sim}$ has infinitely many equivalence classes.

  $[ab] \neq [ab^2] \neq [ab^3] \neq \ldots$

Myhill-Nerode Theorem

Theorem 9 (Myhill-Nerode Theorem)

$\mathcal{L} \subseteq \Sigma^*$ is regular iff $\overset{\mathcal{L}}{\approx}$ finitely many equivalence classes.

Hence

- $\mathcal{L}_1 = \{w \in \{0,1\}^* : \#_1(w) \bmod 4 = 0\}$ is regular.
- $\mathcal{L}_2 = \{0^n 1^n : n \in \mathbb{N}\}$ is not regular.
- $\mathcal{L}_3 = \{a^i b^n c^n : n \geq 0, i \geq 1\} \cup \{b^n c^m : n, m \geq 0\}$ is not regular.

Proving Myhill-Nerode Theorem $\Longrightarrow$

Let $\mathcal{L}$ be a regular language and let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting it.

- Define the binary relation $\overset{M}{\sim}$ by $x \overset{M}{\sim} y$ if $\widehat{\delta}(q_0, x) = \widehat{\delta}(q_0, y)$.
- $\overset{M}{\sim}$ is an equivalence relation.
- $x \overset{M}{\sim} y \implies xz \overset{M}{\sim} yz$ for every $z \in \Sigma^*$.

  $\implies xz \in \mathcal{L}$ iff $yz \in \mathcal{L}$.
- Hence, $x \overset{M}{\sim} y \implies x \overset{\mathcal{L}}{\sim} y$.
- Each equivalence class of $\overset{\mathcal{L}}{\sim}$ corresponds to union of classes of $\overset{M}{\sim}$. Namely, $\overset{M}{\sim}$ is a refinement of $\overset{\mathcal{L}}{\sim}$. (see drawing on board)
- Specifically, # of equivalence classes of $\overset{\mathcal{L}}{\sim}$ is less or equal than # of equivalence classes of $\overset{M}{\sim}$.
- $\overset{M}{\sim}$ has finitely many equivalence classes. (?)
- Therefore, $\overset{\mathcal{L}}{\sim}$ has finitely many equivalence classes. ♠

Proving Myhill-Nerode theorem $\Longleftarrow$

Assume $\overset{\mathcal{L}}{\sim}$ has finitely many equivalence classes and let $x_1, \ldots, x_n \in \Sigma^*$ be their representatives.
We'll construct a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that accepts $\mathcal{L}$.
For $x \in \Sigma^*$, let $C(x)$ be the index $i \in \{1, \ldots, n\}$ with $x \in [x_i]$.

- $Q = \{1, \ldots, n\}$.
- $\delta(i, \sigma) = C(x_i\sigma)$.
- $q_0 = C(\varepsilon)$.
- $F = \{i : x_i \in \mathcal{L}\}$.

Claim. For any $y \in \Sigma^*$: $\quad \widehat{\delta}(q_0, y) = C(y)$.

Hence, $y \in \mathcal{L}(M) \Leftrightarrow \widehat{\delta}(q_0, y) \in F \Longleftrightarrow C(y) \in F \Longleftrightarrow x_{C(y)} \in \mathcal{L} \Longleftrightarrow y \in \mathcal{L}$. $\square$
This is the optimal DFA, number of states wise, for $\mathcal{L}$. (?)
Proof: (of claim) By induction on word length.

- Let $y = w\sigma \in \Sigma^*$, and assume $w \in [x_i]$ and $y \in [x_j]$.
- $\delta(i, \sigma) := C(x_i\sigma) =$ (by right invariance) $C(w\sigma) = C(y) = j$.

$\Longrightarrow$ $\widehat{\delta}(q_0, w\sigma) := \delta(\widehat{\delta}(q_0, w), \sigma) =$ (by i.h) $\delta(i, \sigma) = j$.

Example

Construct a DFA for $\{w \in \{0,1\}^* : \#_1(w) \equiv 0 \pmod 5\}$, via the latter proof method.

- Equivalent class representatives: $\{\varepsilon, 1, 11, 111, 1111\}$
- $Q = \{0, 1, 2, 3, 4\}$
- $q_0 = 0$
- $F = \{0\}$
- $\delta(i, 0) = C(1^i 0) = i$ and $\delta(i, 1) = C(1^i 1) = i + 1 \pmod 5$

Part III

Closure Properties of Regular Languages

Simple closure properties

- Regular languages are closed under complement.
  1. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that accepts $\mathcal{L}$.
  2. Then $M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$ is a DFA that accepts $\overline{\mathcal{L}} = \Sigma^* \setminus \mathcal{L}$.
  3. NFA ?!
- Regular languages are closed under intersection.
  1. $\mathcal{L}_1 \cap \mathcal{L}_2 = \overline{\overline{\mathcal{L}_1} \cup \overline{\mathcal{L}_2}}$.
  2. Proof with automata ?

Division

For languages $\mathcal{L}_1, \mathcal{L}_2 \subseteq \Sigma^*$, define

$$\mathcal{L}_1/\mathcal{L}_2 = \{x \in \Sigma^* : \exists y \in \mathcal{L}_2,\ xy \in \mathcal{L}_1\}$$

Examples:

- $\mathcal{L}_1 = \mathcal{L}(01 \cup 1)^*$ and $\mathcal{L}_2 = \mathcal{L}(00)$. Then $\mathcal{L}_1/\mathcal{L}_2 = \emptyset$
- $\mathcal{L}_3 = \mathcal{L}(a^*b^*c^*)$ and $\mathcal{L}_4 = \mathcal{L}(b)$. Then $\mathcal{L}_3/\mathcal{L}_4 = \mathcal{L}(a^*b^*)$

Closure under division

Recall, $\mathcal{L}_1/\mathcal{L}_2 = \{x : \exists y \in \mathcal{L}_2, xy \in \mathcal{L}_1\}$

Theorem 10

Regular languages are closed under division with any language :$\mathcal{L}_1$ is regular $\implies$ $\mathcal{L}_1/\mathcal{L}_2$ is regular.

Proof: Let $\mathcal{L}_1$ be a regular language and let $\mathcal{L}_2$ be an arbitrary language.

- $\overset{\mathcal{L}_1}{\sim}$ is a refinement of $\overset{\mathcal{L}_1/\mathcal{L}_2}{\sim}$. Proof: Assume $x \overset{\mathcal{L}_1}{\sim} y$. For $z \in \Sigma^*$:

  $xz \in \mathcal{L}_1/\mathcal{L}_2 \leftrightarrow xzw \in \mathcal{L}_1 \leftrightarrow yzw \in \mathcal{L}_1 \leftrightarrow yz \in \mathcal{L}_1/\mathcal{L}_2$. $\implies$ $x \overset{\mathcal{L}_1/\mathcal{L}_2}{\sim} y$. □
- Hence, $\mathcal{L}_1/\mathcal{L}_2$ has finite number of equivalent states, and thus regular.
- Another proof.
- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for $\mathcal{L}_1$.
- Let $F' = \{q \in Q : \exists y \in \mathcal{L}_2, \widehat{\delta}(q, y) \in F\}$
- The DFA $M' = (Q, \Sigma, \delta, q_0, F')$ accepts $\mathcal{L}_1/\mathcal{L}_2$. □

F' is well defined, but might be hard to compute – "non constructive proof".

Homomorphism

Definition 11 (Homomorphism)

An homomorphism from alphabet Δ to words over alphabet Σ, is a function $h\colon \Delta \mapsto \Sigma^*$.
For $w \in \Delta^*$, let $h(w = w_1, \ldots, w_n) = h(w_1) \cdots h(w_n)$.
For $\mathcal{L} \subseteq \Delta^*$, let $h(\mathcal{L}) = \{h(w)\colon w \in \mathcal{L}\}$.

By definition, $h(\varepsilon) = \varepsilon$ and $h(\emptyset) = \emptyset$.

Examples:

- Let $h\colon \{0,1\} \mapsto \{a,b\}^*$ be defined by $h(1) = aba$ and $h(0) = aa$.

  $h(010) = aa\,aba\,aa$. For $\mathcal{L}_1 = (01)^*$, $h(\mathcal{L}_1) = (aaaba)^*$.
- Let $h(0) = a$, $h(1) = a$. For $\mathcal{L}_2 = \{0^n 1^n\colon n \geq 0\}$, $h(\mathcal{L}_2) = \{a^{2n}\colon n \geq 0\}$.

Closure under homomorphism

Theorem 12

Regular languages are closed under homomorphism.

Proof idea: Using regular expressions.

Let $\mathcal{L} \subseteq \Delta^*$ be regular language and let $h\colon \Delta \mapsto \Sigma^*$.

1. $h(\mathcal{L}(\emptyset)) = \emptyset$, $h(\mathcal{L}(\varepsilon)) = \{\varepsilon\}$, and $h(\mathcal{L}(a)) = \{h(a)\}$ for any $a \in \Delta$.
2. $h(\mathcal{L}_1 \cup \mathcal{L}_2) = h(\mathcal{L}_1) \cup h(\mathcal{L}_2)$, $h(\mathcal{L}_1 \| \mathcal{L}_2) = h(\mathcal{L}_1) \| h(\mathcal{L}_2)$ and $h(\mathcal{L}^*) = h(\mathcal{L})^*$

Let R be a regular expression with $\mathcal{L}(R) = \mathcal{L}$. The proof is by induction on $|R|$.

- $|R| = 1$. By Item 1, $h(\mathcal{L}) = h(\mathcal{L}(R))$ is regular.
- $|R| > 1$. Assume $R = (R_1 \cup R_2)$ (other cases are similar).
  - By item 2, $h(\mathcal{L}) = h(\mathcal{L}(R_1) \cup \mathcal{L}(R_2)) = h(\mathcal{L}(R_1)) \cup h(\mathcal{L}(R_2))$.
  - By i.h., $h(\mathcal{L}(R_1)$ and $h(\mathcal{L}(R_2)$ are regular.

  Thus, $h(\mathcal{L})$ is regular. $\square$

Closure under homomorphism, proof using Automata

Let $M = (Q, \Delta, \delta, q_0, F)$ bea DFA for $\mathcal{L}$.

Define **NFA** $N = (Q', \Sigma, \delta', q_0, F)$ for $h(\mathcal{L})$ as follows:

- if $\delta(q_i, \sigma) = q_j$ and $h(\sigma) = w_1, \ldots, w_t$, then $\delta'(d^k_{i\sigma}, w_i) = d^{k+1}_{i\sigma}$ for all $k \in \{1, \ldots, t-1\}$, letting $d^1_{i\sigma} = q_i$ and $d^t_{i\sigma} = q_j$.
- Q' includes Q and all new states.

Claim 13

$\mathcal{L}(N) = h(\mathcal{L})$.

Proof idea:

$h(\mathcal{L}) \subseteq \mathcal{L}(N)$: $w \in \mathcal{L} \implies \exists r_1, \ldots, r_{|w|+1}$ s.t. $r_1 = q_0$, $r_{|w|+1} \in F$ and $r_{i+1} = \delta(r_i, w_{i+1})$. $\ldots \implies h(w) \in \mathcal{L}(N)$.

$\mathcal{L}(N) \subseteq h(\mathcal{L})$: $w \in \mathcal{L}(N) \implies \exists r_1, \ldots, r_{|w|+1}$ s.t. $r_1 = q_0$, $r_{|w|+1} \in F$ and $r_{i+1} \in \delta'(r_i, w_{i+1})$. $\ldots \implies \exists w' \in \mathcal{L}$ with $h(w') = w$.

Inverse homomorphism

Definition 14 (Inverse homomorphism)

For homomorphism $h\colon \Delta \mapsto \Sigma^*$, define its inverse homomorphism $h^{-1}\colon \Sigma^* \mapsto P(\Delta^*)$, by $h^{-1}(w) = \{x \in \Delta^* \colon h(x) = w\}$.

For $\mathcal{L} \subseteq \Sigma^*$, let $h^{-1}(\mathcal{L}) = \bigcup_{x\in\mathcal{L}} h^{-1}(x) = \{x \in \Delta^* \colon h(x) \in \mathcal{L}\}$

- Example: $h(0) = a$, $h(1) = b$ and $h(2) = a$. Then $h^{-1}(\{a^n b a^n \colon n \geq 0\}) = \{\{0,2\}^n 1 \{0,2\}^n \colon n \geq 0\}$.

Claim 15

For any $h\colon \Delta \mapsto \Sigma^*$:

1. $h(h^{-1}(\mathcal{L})) \subseteq \mathcal{L}$, for any $\mathcal{L} \subseteq \Sigma^*$
2. $\mathcal{L} \subseteq h^{-1}(h(\mathcal{L}))$, for any $\mathcal{L} \subseteq \Delta^*$

Proof:

1. Immediate
2. Holds since $w \in h^{-1}(h(w))$ for any $w \in \Delta^*$

Closure under inverse homomorphism

Theorem 16

Regular languages are closed under inverse homomorphism.

Proof idea: Let $\mathcal{L}$ be a regular language, let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for $\mathcal{L}$ and let $h\colon \Delta \mapsto \Sigma^*$.

- $w \in h(\mathcal{L}) \longleftrightarrow h(w) \in \mathcal{L}(M)$.
- Hence, to decide w simply emulate $M(h(w))$...

More formally,

- Let $M' = (Q, \Delta, \delta', q_0, F)$ with $\delta'(q, a) = \widehat{\delta}(q, h(a))$.
- Easy to verify that $\widehat{\delta'}(q, w) = \widehat{\delta}_M(q, h(w))$

$\Longrightarrow$ $w \in \mathcal{L}(M') \longleftrightarrow h(w) \in \mathcal{L}(M) \longleftrightarrow w \in h^{-1}(\mathcal{L})$ $\square$

Part IV

Algorithmic Questions for NFAs

Algorithmic Questions for NFAs

Q: Given an NFA, N, and a string w, is $w \in \mathcal{L}(N)$?

Answer: Construct the DFA equivalent to N and run it on w.

Q: Is $\mathcal{L}(N) = \emptyset$?

Answer: This is a reachability question in graphs: Is there a path in the states' graph of N from the start state to some accepting state?

There are simple, efficient algorithms for this task.

More Questions

Q.: Is $\mathcal{L}(N) = \Sigma^*$?

Answer: Check if $\overline{\mathcal{L}(N)} = \emptyset$.

Q.: Given N_1 and N_2, is $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$?

Answer: Check if $\overline{\mathcal{L}(N_2)} \cap \mathcal{L}(N_1) = \emptyset$.

Q.: Given N_1 and N_2, is $\mathcal{L}(N_1) = \mathcal{L}(N_2)$?

Answer: Check if $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$ and $\mathcal{L}(N_2) \subseteq \mathcal{L}(N_1)$.

In the future, we will see that for stronger models of computations, many of these problems cannot be solved by any algorithm.

Part V

Summary — Regular Languages

Summary - Regular Languages

So far we saw

- Finite automata,
- Regular languages,
- Regular expressions,
- Myhill-Nerode theorem and pumping lemma for regular languages.
- Did not do (see appendix): Given a DFA M, the minimal (in $\#$ of states) DFA M' with $\mathcal{L}(M') = \mathcal{L}(M)$.

Next class we introduce stronger machines and languages with more expressive power:

- pushdown automata,
- context-free languages,
- context-free grammars

Part VII

Appendix
Not Taught in Class

Section 1

Finding the Minimal Automata

Finding the minimal automata

Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, find the minimal (with respect to $\#$ of states) DFA M' with $\mathcal{L}(M') = \mathcal{L}(M)$.

Definition 17 (Equivalent states)

For DFA $(Q, \Sigma, \delta, q_1, F)$, states $q_1, q_2 \in Q$ are equivalent, if for all $x_1, x_2 \in \Sigma^*$ with $\widehat{\delta}(q_0, x_i) = q_i$, it holds that $x_1 \overset{\mathcal{L}}{\sim} x_2$.

Idea: keep merging equivalent states in Q, until all states are non-equivalent.

Actual idea:

1. Start with the two sets F and $Q \setminus F$.
2. Keep *splitting* the sets until all states in the same set are equivalent.

   To check whether states q and q' are equivalent, check if $\delta(q, \sigma)$ and $\delta(q', \sigma)$ are in the same set, for all $\sigma \in \Sigma$.(?)
3. Merge all states in the same set.

Is this a minimal automate? Yes, if M has no unreachable states.

But we can assume that wlg. (?)

Finding the minimal automata

Algorithm 18

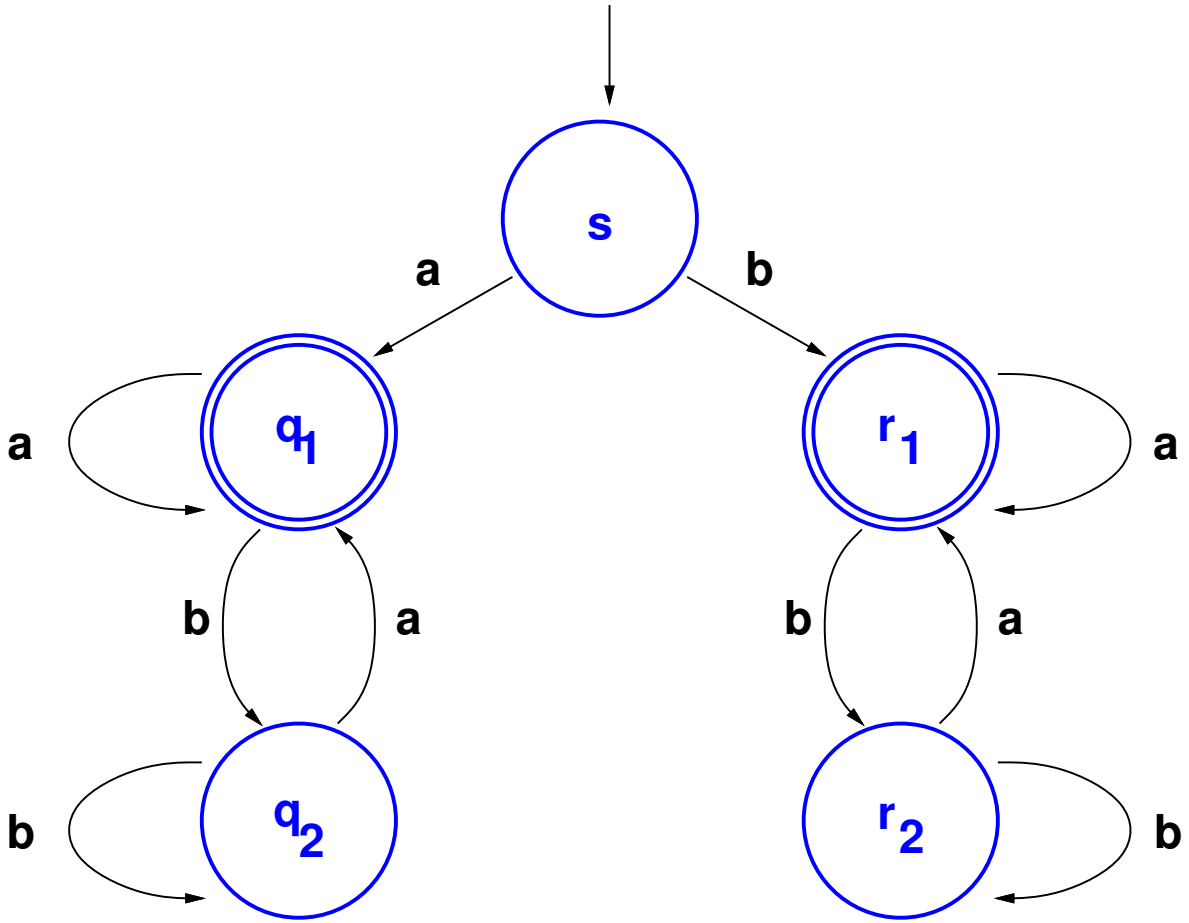
Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

1. Let $\mathcal{T} = \{F, Q \setminus F\}$.
2. While $\exists\, \mathcal{S} \in \mathcal{T}$, $q_1, q_2 \in \mathcal{S}$ and $\sigma \in \Sigma^*$ s.t, $\delta(q_1, \sigma) \in \mathcal{S}'$ and $\delta(q_2, \sigma) \notin \mathcal{S}'$ for some $\mathcal{S}' \in \mathcal{T}$:
   - **2.1** Let $\mathcal{S}_{sp} = \{q \in \mathcal{S} : \delta(q, \sigma) \in \mathcal{S}'\}$.
   - **2.2** Set $\mathcal{T} = \mathcal{T} \cup \mathcal{S}_{sp} \cup (\mathcal{S} \setminus \mathcal{S}_{sp}) \setminus \mathcal{S}$.
3. Output DFA $M' = (Q', \delta', q_0', F')$, where
   - $Q' = \mathcal{T}$
   - $q_0' = \mathcal{S}_0 \in \mathcal{T}$, where $q_0 \in \mathcal{S}_0$.
   - $F' = \{\mathcal{S} \in \mathcal{T} : \mathcal{S} \subseteq F\}$
   - $\delta'(\mathcal{S}, \sigma) = \mathcal{S}' \in \mathcal{T}$, s.t. $\delta(q, \sigma) \in \mathcal{S}'$ for (every) $q \in \mathcal{S}$.

Claim 19

The above algorithm outputs the minimal automata for $\mathcal{L}(M)$.

Example



On board...

Section 2

Using Homomorphism

Using homomorphism

We know that $\mathcal{L}_1 = \{0^n 1^n : n \geq 1\}$ is not regular, show that $\mathcal{L}_2 = \{a^n b a^n : n \geq 1\}$ is not regular.

We will prove using homomorphism and inverse homomorphism. Let

- $h_1(a) = a$, $h_1(b) = b$, $h_1(c) = a$. $(h_1 : \{a, b, c\} \mapsto \{a, b, c\}^*)$
- $h_2(a) = 0$, $h_2(b) = \epsilon$, $h_2(c) = 1$. $(h_1 : \{a, b, c\} \mapsto \{0, 1\}^*)$

We prove $h_2(h_1^{-1}(\mathcal{L}_2) \cap a^* b^* c^*) = \mathcal{L}_1$. Thus, $\mathcal{L}_2$ is not regular (?)

- $h_1^{-1}(\mathcal{L}_2) = \mathcal{L}((a \cup c)^n)\ b\ \mathcal{L}((a \cup c)^n)$
- $h_1^{-1}(\mathcal{L}_2) \cap a^* b^* c^* = \{a^n b c^n : n \geq 1\}$
- $h_2(h_1^{-1}(\mathcal{L}_2) \cap a^* b^* c^*) = \{0^n 1^n : n \geq 1\}$