Exercise 4 - Computational Models

1. Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Give a formal description of a TM that accepts $L(A)$. describe formally each one of the elements in $M = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$.

2. Let $\text{Prefix}(L) = \{x \mid \exists y \in \Sigma^* \text{ such that } xy \in L\}$. Prove that $\mathcal{RE}$ is closed under Prefix.

3. A function $f : \Sigma^* \to \Gamma^*$ is computable if there exists a TM that for every $x \in \Sigma^*$ halts with $f(x)$ on its tape, when given x as input. For a function $f : \Sigma^* \to \Gamma^*$ let language $L_f = \{(x, f(x)) | x \in \Sigma^*\}$.

   Show that $L_f \in \mathcal{RE} \iff f$ is computable.

4. Prove or disprove:

   (a) R is closed under complementation.

   (b) $R.E.$ is closed under complementation.

   (c) $R.E.$ is closed under intersection.

   (d) $co - R.E.$ is closed under intersection.

   (e) $R.E$ is closed under Kleene star.

5. Let $L = \{< M_1, M_2 > |L(M_1) \cap L(M_2) = \emptyset\}$ ($< M_1, M_2 >$ represent an encoding of 2 Turing Machines). Prove $L \in co - \mathcal{RE}$.

6. A researcher claims that he has detected for every alpahbet Σ a finite set of important TMs that can decide any decidable language over Σ.

   (a) Prove that he is wrong.

   (b) Let A be the set of languages decided by TMs that have at most 1000 states and at most 1000 tape symbols (in addition to the symbols of Σ and $\sqcup$). Prove that $A \neq R$.

7. The $3x + 1$ problem concerns the function f, which takes odd integers n to $3n + 1$ and even integers n to $n/2$. If you start with an integer n and iterate f, you obtain a sequence x, f(x), f(f(x)),... Stop if you ever hit 1. For example, if $n = 17$, you get the sequence 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Extensive computer tests have shown that every starting point between 1 and a large integer gives a sequence that ends in 1. But, the question of whether all positive starting points end up at 1 is unresolved; it is called the $3x + 1$ problem. Suppose that A_{TM} were decidable by a TM H. Use H to describe a TM that prints the answer to the $3x + 1$ problem.