Solution sketch 3 - Computational Models - Fall 2017

1. The language of the grammar consists of the set $L(G) = \{(ab)^n(cb^+)^n | n \geq 0\}$. The recursive S rule generates an equal number of ab's and cB's. The B rules generate b^+. In a derivation, each occurrence of B may produce a different number of bs.

2. The grammar is ambiguous because we can find strings which have multiple derivations:

$$S \Rightarrow 0A \Rightarrow 00AA \Rightarrow 001S1 \Rightarrow 0011B1 \Rightarrow 001101$$
$$S \Rightarrow 0A \Rightarrow 00AA \Rightarrow 0011S \Rightarrow 00110A \Rightarrow 001101$$

3. (a) $G = (\{S, A, B, C, D, E, F\}, \{0, 1, 2\}, R, S)$ where R contains the rules

$$S \rightarrow AB \mid BC \mid DE \mid DF$$
$$A \rightarrow 0 \mid 0A \mid 0A1$$
$$B \rightarrow 1 \mid B1 \mid 0B1$$
$$C \rightarrow 2 \mid 2C$$
$$D \rightarrow 0 \mid 0D$$
$$E \rightarrow 1 \mid 1E \mid 1E2$$
$$F \rightarrow 2 \mid F2 \mid 1F2$$

   (b) $G = (\{S\}, \{0, 1\}, R, S)$ where R contains the rules

$$S \rightarrow 1S0S0S \mid 0S1S0S \mid 0S0S1S \mid \varepsilon$$

4. Assume to the contrary that L is context-free and let $l > 0$ be the promised pumping constant. Take $w = a^l b^l c^l d^l \in L$ and obviously $|w| \geq l$. Consider a

general decomposition $w = uvxyz$ such that $|vy| > 0$ and $|vxy| \geq l$. We shall see that in any such decomposition, $w' = uv^0xy^0z \notin L$, in contradiction to the Pumping Lemma. In every such decomposition vxy satisfies one of the following:

(a) vxy contains only a-s, so $vy = a^s$ for $s > 0$.

(b) vxy contains only b-s, so $vy = b^s$ for $s > 0$.

(c) vxy contains only c-s, so $vy = c^s$ for $s > 0$.

(d) vxy contains only d-s, so $vy = d^s$ for $s > 0$.

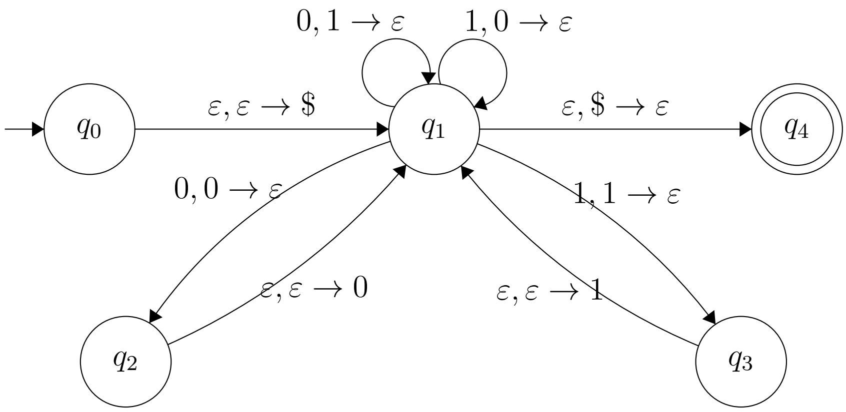
(e) vxy contains only a-s and b-s, so $vy = a^sb^t$ for $s + t > 0$.

(f) vxy contains only b-s and c-s, so $vy = b^sc^t$ for $s + t > 0$.

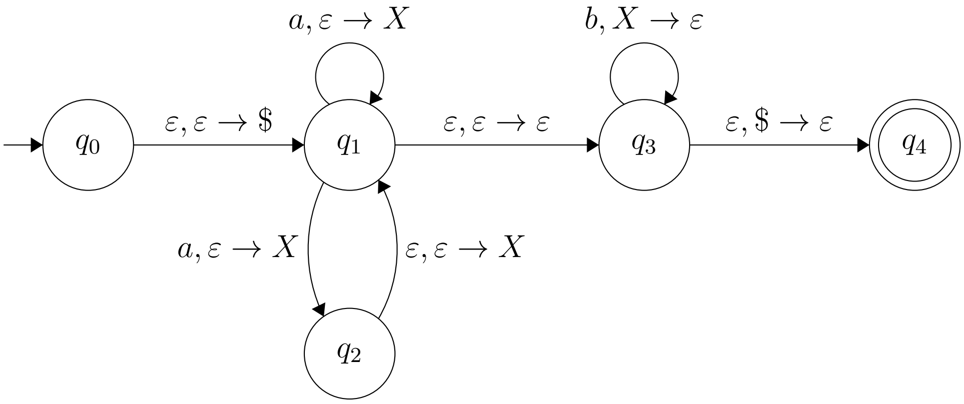
(g) vxy contains only c-s and d-s, so $vy = c^sd^t$ for $s + t > 0$.

In form (a) (and likewise for the following three), $w' = a^{l-s}bc^ld^l$. As $s > 0$, $w' \notin L$. In form (e) (and likewise for the following two), $w' = a^{l-s}b^{l-t}c^ld^l$. As $s+t \geq 1$ then either $s \geq 1$ or $t \geq 1$, so either $\#_a(w') = l - s < l = \#_c(w)$ or $\#_b(w) = l - t < l = \#_d(w)$. In any case, $w' \notin L$.

5. (a) $L = \{w \mid \#_0(w) = \#_1(w)\}$ over $\Sigma = \{0, 1\}$



(b) $L = \{a^ib^j | j \geq 0, i \leq j \leq 2i\}$ over $\Sigma = \{a, b\}$



6. The claim is incorrect. Let $A = \{0^n 1^{2n} \mid n > 0\}$ and $b = \{0^{2n} 1^n \mid n > 0\}$. Both are CFL (why?). The $perfect\ shuffle$ of A and B is $\{0^n (10)^n 1^n \mid n > 0\}$ which is not CFL (proof by pumping lemma, similar to the proof of $\{a^n b^n c^n \mid n > 0\}$).

7. The claim is correct. Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that accepts L_r. The idea is to construct a PDA that consists of two copies of M. The first copy pushes a symbol X to the stack after every character read and the second pops. We can always “guess” which s takes us from the first copy to the second. We accept iff we are in an accepting state of M in the second copy and the stack is empty. Formally, construct a PDA

$$M' = (Q \times \{1, 2\} \cup q_i, q_a, \Sigma, \{\$, X\}, \delta', q_i, \{q_a\})$$

where:

- $\delta'(q_i, \varepsilon, \varepsilon) = \{((q_0, 1), \$)\}$
- $\forall q \in Q \forall \sigma \in \Sigma\ \delta'((q, 1), \sigma, \varepsilon) = \{((\delta(q, \sigma), 1), X), ((q, 2), \varepsilon)\}$
- $\forall q \in Q \forall \sigma \in \Sigma\ \delta'((q, 2), \sigma, X) = \{((\delta(q, \sigma), 2), \varepsilon)\}$
- $\forall q \in F\ \delta'((q, 2), \sigma, \$) = \{(q_a, \varepsilon)\}$

For correctness we give the first direction (the second direction is almost identical):

$$
\begin{aligned}
x \in L' &\Rightarrow \exists u, \sigma, v \; uv \in L_r \wedge \sigma \in \Sigma \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v \; uv \in L(M) \wedge \sigma \in \Sigma \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v, q, q' \; \hat{\delta}(q_0, u) = q \wedge \hat{\delta}(q, v) = q' \wedge q' \in F \wedge \sigma \in \Sigma \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v, q, q' \; ((q_0, 1), \$) \in \delta'(q_i, \varepsilon, \varepsilon) \wedge ((q, 1), \$X^{|u|}) \in \hat{\delta}'((q_0, 1, u, \$) \wedge \\
&\quad ((q, 2, \varepsilon) \in \hat{\delta}'((q, 1), \sigma, \varepsilon) \wedge ((q', 2), \$X^{|u|-|v|}) \in \hat{\delta}'((q, 2), v, \$X^{|u|}) \wedge q' \in F \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v, q, q' \; ((q_0, 1), \$) \in \delta'(q_i, \varepsilon, \varepsilon) \wedge ((q, 1), \$X^{|u|}) \in \hat{\delta}'((q_0, 1, u, \$) \wedge \\
&\quad ((q, 2, \varepsilon) \in \hat{\delta}'((q, 1), \sigma, \varepsilon) \wedge ((q', 2), \$) \in \hat{\delta}'((q, 2), v, \$X^{|u|}) \wedge (q_a, \varepsilon) \in \hat{\delta}'((q, 2), \varepsilon, \$) \\
&\Rightarrow \exists u, v \; (q_a, \varepsilon) \in \hat{\delta}'((q_i, uv), \varepsilon) \\
&\Rightarrow \exists x \in L(M')
\end{aligned}
$$

Note that the claims $((q, 1), \$X^{|u|}) \in \hat{\delta}'((q_0, 1), u, \$)$ and $((q', 2), \$X^{|u|-|v|}) \in \hat{\delta}'((q, 2), v, \$X^{|u|})$ require an induction on the length of u and v.